Homework

1. Find the inverse of the following matrix if it is invertible

(1) $A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 2 \\ 1 & 3 & 4 \end{pmatrix}$ (2) $A = \begin{pmatrix} 2 & 5 & 0 & 0 \\ 1 & 3 & 0 & 0 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 2 & 5 \end{pmatrix}$ (2) $A = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 & 0 \\ 1 & 3 & 3 & 1 & 0 \\ 1 & 4 & 6 & 4 & 1 \end{pmatrix}$

2. For which values of the constant k is the matrix $A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & k \\ 1 & 4 & k^2 \end{pmatrix}$ invertible?

3. Find all matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ such that $ad - bc = 1$ and $A^{-1} = A$

4. Let $A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 2 & 1 \\ 3 & 4 & 3 \end{pmatrix}, B = \begin{pmatrix} 2 & 1 \\ 5 & 3 \end{pmatrix}, C = \begin{pmatrix} 1 & 3 \\ 2 & 0 \\ 3 & 1 \end{pmatrix}$.

Find a matrix X satisfying $AXB = C$.

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5. Let $P = \begin{pmatrix} 1 & 2 \\ 1 & 4 \end{pmatrix}$, $\Lambda = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$, $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. If $AP = P\Lambda$,
Compute (1) A^n (2) $A^2 - 3A + 2I$

6. Let $A = \begin{bmatrix} 3 & 1 \\ 3 & 5 \end{bmatrix}$ in all parts of this problem.

a. Find a scalar λ (lambda) such that the matrix $A-\lambda I_2$ fails to be invertible. There are two solutions; choose one and use it in parts (b) and (c).

b. For the λ you chose in part (a), find the matrix $A - \lambda I_2$; then find a nonzero vector $\vec{x}$ such that $(A - \lambda I_2)\vec{x} = \vec{0}$. (This can be done, since $A - \lambda I_2$ fails to be invertible.)

c. Note that the equation $(A - \lambda I_2)\vec{x} = \vec{0}$ can be written as $A\vec{x} - \lambda\vec{x} = \vec{0}$, or $A\vec{x} = \lambda\vec{x}$. Check that the equation $A\vec{x} = \lambda\vec{x}$ holds for your λ from part (a) and your $\vec{x}$ from part (b).

Homework

7. True or false

(1) If A and B are two $n \times n$ invertible matrices, then $A + B$ is invertible and

$$(A+B)^{-1} = A^{-1} + B^{-1}$$

(2) If the homogeneous system $Ax = O$ has a nonzero solution, A is not invertible.

(3) Diagonally dominant matrices ($|a_{ii}| > \sum_{j \neq i} |a_{ij}|$ for $1 \leq i \leq n$) are invertible.

(4) Let A be an invertible lower (upper) triangular square matrix. Then its inverse A^{-1} is an invertible lower (upper) triangular matrix

8. Find the inverse of A. ($a_1 a_2 \cdots a_n \neq 0, n \geq 2$)

$$A = \begin{pmatrix} 0 & a_1 & 0 & \cdots & 0 \\ 0 & 0 & a_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & a_{n-1} \\ a_n & 0 & 0 & \cdots & 0 \end{pmatrix}_{n \times n}$$

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$$\left[\begin{array}{cc:cc} a & b & 1 & 0 \\ c & d & 0 & 1 \end{array}\right] \quad \div a, \text{ requires 2 multiplicative operations: } b/a \text{ and } 1/a$$

$$\downarrow$$

$$\left[\begin{array}{cc:cc} 1 & b' & e & 0 \\ c & d & 0 & 1 \end{array}\right] \quad \begin{array}{l} (\text{where } b' = b/a, \text{ and } e = 1/a) \\ -c\ (\mathrm{I}), \text{ requires 2 multiplicative operations: } cb' \text{ and } ce \end{array}$$

$$\downarrow$$

$$\left[\begin{array}{cc:cc} 1 & b' & e & 0 \\ 0 & d' & g & 1 \end{array}\right] \quad \div d', \text{ requires 2 multiplicative operations}$$

$$\downarrow$$

$$\left[\begin{array}{cc:cc} 1 & b' & e & 0 \\ 0 & 1 & g' & h \end{array}\right] \quad -b'\ (\mathrm{II}), \text{ requires 2 multiplicative operations}$$

$$\downarrow$$

$$\left[\begin{array}{cc:cc} 1 & 0 & e' & f \\ 0 & 1 & g' & h \end{array}\right]$$

9. To gauge the complexity of a computational task, mathematicians and computer scientists count the number of elementary operations (additions, subtractions, multiplications, and divisions) required. For a rough count, we will sometimes consider multiplications and divisions only, referring to those jointly as *multiplicative operations*. As an example, we examine the process of inverting a 2×2 matrix by elimination.

The whole process requires eight multiplicative operations. Note that we do not count operations with predictable results, such as $1a$, $0a$, a/a, $0/a$.

a. How many multiplicative operations are required to invert a 3×3 matrix by elimination?

b. How many multiplicative operations are required to invert an $n \times n$ matrix by elimination?

c. If it takes a slow hand-held calculator 1 second to invert a 3×3 matrix, how long will it take the same calculator to invert a 12×12 matrix? Assume that the matrices are inverted by Gauss–Jordan elimination and that the duration of the computation is proportional to the number of multiplications and divisions involved.

Homework (Optional)

1. Find the inverse of the following matrix if it is invertible

(1) $A = \begin{pmatrix} 0 & 1 & 1 & \cdots & 1 \\ 1 & 0 & 1 & \cdots & 1 \\ 1 & 1 & 0 & \cdots & 1 \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & 1 & 1 & \cdots & 0 \end{pmatrix}_{n\times n}$ (2) $A = \begin{pmatrix} 1 & a & a^2 & \cdots & a^{n-1} \\ 0 & 1 & a & \cdots & a^{n-2} \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & a \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix}_{n\times n}$

2. Solve the matrix equation

$$\begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ 0 & 1 & 1 & \cdots & 1 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix}_{n\times n} X = \begin{pmatrix} 1 & 2 & 3 & \cdots & n \\ 0 & 1 & 2 & \cdots & n-1 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 2 \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix}_{n\times n}$$

3. Let $A_{n\times m}$, $B_{m\times n}$ be two matrices.

Prove or disprove: if $I_n - AB$ is invertible, then $I_m - BA$ is also invertible.

Homework (Optional)

4.Consider the matrix

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 6 & 7 \\ 2 & 2 & 4 \end{bmatrix}.$$

a. Find lower triangular elementary matrices E_1, $E_2, \ldots, E_m$ such that the product

$$E_m \cdots E_2 E_1 A$$

is an upper triangular matrix U. *Hint*: Use elementary row operations to eliminate the entries below the diagonal of A.

b. Find lower triangular elementary matrices M_1, $M_2, \ldots, M_m$ and an upper triangular matrix U such that

$$A = M_1 M_2 \cdots M_m U.$$

c. Find a lower triangular matrix L and an upper triangular matrix U such that

$$A = LU.$$

Such a representation of an invertible matrix is called an *LU-factorization*. The method outlined in this exercise to find an LU-factorization can be streamlined somewhat, but we have seen the major ideas. An LU-factorization (as introduced here) does not always exist.

d. Find a lower triangular matrix L with 1's on the diagonal, an upper triangular matrix U with 1's on the diagonal, and a diagonal matrix D such that $A = LDU$. Such a representation of an invertible matrix is called an *LDU-factorization*.

Homework (Optional)

5. (a) Find all invertible $n \times n$ matrices A such that $A^2 = A$.

(b) Find a nonzero $n \times n$ matrix A with identical entries such that $A^2 = A$.

6. Consider two $n \times n$ matrices A and B whose entries are positive or zero. Suppose that all entries of A are less than or equal to s, and all column sums of B are less than or equal to r (the jth column sum of a matrix is the sum of all the entries in its jth column). Show that all entries of the matrix AB are less than or equal to sr.

7. (This exercise builds on Exercise 6. .) Consider an $n \times n$ matrix A whose entries are positive or zero. Suppose that all column sums of A are less than 1. Let r be the largest column sum of A.

a. Show that the entries of A^m are less than or equal to r^m, for all positive integers m.

b. Show that

$$\lim_{m \to \infty} A^m = 0$$

(meaning that all entries of A^m approach zero).

c. Show that the infinite series

$$I_n + A + A^2 + \cdots + A^m + \cdots$$

converges (entry by entry).

d. Compute the product

$$(I_n - A)(I_n + A + A^2 + \cdots + A^m).$$

Simplify the result. Then let m go to infinity, and thus show that

$$(I_n - A)^{-1} = I_n + A + A^2 + \cdots + A^m + \cdots.$$