Homework

1. Compute the determinants using a cofactor expansion

(1) $\begin{vmatrix} -3 & 4 & 2 \\ 6 & 3 & 1 \\ 4 & -7 & -8 \end{vmatrix}$ (across the Third Column)

(2) $\begin{vmatrix} -2 & 4 & 7 & 1 \\ 3 & 0 & 0 & 0 \\ 8 & 5 & 10 & 5 \\ 6 & 0 & 5 & 0 \end{vmatrix}$ (a)across the Second Row (b)across the Fourth Column

2. Compute the determinant: (1) $\begin{vmatrix} 3 & 2 & 4 & -1 & 5 \\ -2 & 0 & 1 & 3 & 2 \\ 1 & 0 & 0 & 4 & 0 \\ 6 & 0 & 2 & -1 & 0 \\ 3 & 0 & 5 & 1 & 0 \end{vmatrix}$ (2) $\begin{vmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 2 & 2 & 2 \\ 1 & 1 & 3 & 3 & 3 \\ 1 & 1 & 1 & 4 & 4 \\ 1 & 1 & 1 & 1 & 5 \end{vmatrix}$

Homework

3. Solve the equation for x：

(1) $\begin{vmatrix} 0 & x-1 & 1 \\ x-1 & 0 & x-2 \\ 1 & x-2 & 0 \end{vmatrix} = 0$ (2) $\begin{vmatrix} 2 & x+2 & 6 \\ 1 & x & 3 \\ 1 & 3 & x \end{vmatrix} = 0$

4. Prove or disprove: $\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} = (a-b)(b-c)(c-a)$

5. Consider two distinct real numbers, a and b. We define the function

$$f(t) = \det \begin{bmatrix} 1 & 1 & 1 \\ a & b & t \\ a^2 & b^2 & t^2 \end{bmatrix}.$$

a. Show that $f(t)$ is a quadratic function. What is the coefficient of t^2?

b. Explain why $f(a) = f(b) = 0$. Conclude that $f(t) = k(t-a)(t-b)$, for some constant k. Find k, using your work in part (a).

c. For which values of t is the matrix invertible?

Homework(Optional)

1. Which of the following functions F of $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ are linear in both columns? Which are linear in both rows? Which are alternating on the columns?

   a. $F(A) = bc$ **b.** $F(A) = cd$ **c.** $F(A) = ac$

   d. $F(A) = bc - ad$ **e.** $F(A) = c$

2. Show that the function

$$F \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & j \end{bmatrix} = bfg$$

   is linear in all three columns and in all three rows. See Example 6. Is F alternating on the columns? See Example 4.

3. For a fixed positive integer n, let D be a function which assigns to any $n \times n$ matrix A a number $D(A)$ such that

   a. D is linear in the rows (see Theorem 6.2.2),

   b. $D(B) = -D(A)$ if B is obtained from A by a row swap, and

   c. $D(I_n) = 1$.

   Show that $D(A) = \det(A)$ for all $n \times n$ matrices A. *Hint*: Consider $E = \operatorname{rref} A$. Think about the relationship between $D(A)$ and $D(E)$, mimicking Algorithm 6.2.5.

   The point of this exercise is that $\det(A)$ can be characterized by the three properties a, b, and c; the determinant can, in fact, be *defined* in terms of these properties. Ever since this approach was first presented in the 1880s by the German mathematician Karl Weierstrass (1817–1897), this definition has been generally used in advanced linear algebra courses because it allows a more elegant presentation of the theory of determinants.

Homework(Optional)

4. Consider a skew-symmetric $n \times n$ matrix A, where n is odd. Show that A is noninvertible, by showing that $\det A = 0$.

5. Consider three distinct points $\begin{bmatrix} a_1 \\ a_2 \end{bmatrix}, \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}, \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$ in the plane. Describe the set of all points $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ satisfying the equation

$$\det \begin{bmatrix} 1 & 1 & 1 & 1 \\ x_1 & a_1 & b_1 & c_1 \\ x_2 & a_2 & b_2 & c_2 \\ x_1^2 + x_2^2 & a_1^2 + a_2^2 & b_1^2 + b_2^2 & c_1^2 + c_2^2 \end{bmatrix} = 0.$$

6. Let M_n be the matrix with all 1's along the main diagonal, directly above the main diagonal, and directly below the diagonal, and 0's everywhere else. For example,

$$M_4 = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}.$$

Let $d_n = \det(M_n)$.

a. Find a formula expressing d_n in terms of d_{n-1} and d_{n-2}, for positive integers $n \geq 3$.

b. Find $d_1, d_2, \ldots, d_8$.

c. What is the relationship between d_n and d_{n+3}? What about d_n and d_{n+6}?

d. Find d_{100}.