Homework

5. Please determine if the vertices $A(1,0,2), B(3,-1,1), C(0,-2,-1), D(-1,2,3)$ lie in the same plane and find the volume of the tetrahedra whose vertices are $A(1,0,2), B(3,-1,1), C(0,-2,-1), D(-1,2,3)$.

6. True or false

(1) $\vec{u} \parallel \vec{v} \Leftrightarrow \vec{u} \times \vec{v} = \vec{0}$

(2) $\vec{u} \times (\vec{v} + \vec{w}) = \vec{u} \times \vec{v} + \vec{u} \times \vec{w}$

(3) $\vec{u} \times (\vec{v} \times \vec{w}) = (\vec{u} \cdot \vec{w})\vec{v} - (\vec{v} \cdot \vec{w})\vec{u}$

(4) $(\vec{u} \times \vec{v}) \times \vec{w} = (\vec{u} \cdot \vec{w})\vec{v} - (\vec{u} \cdot \vec{v})\vec{w}$

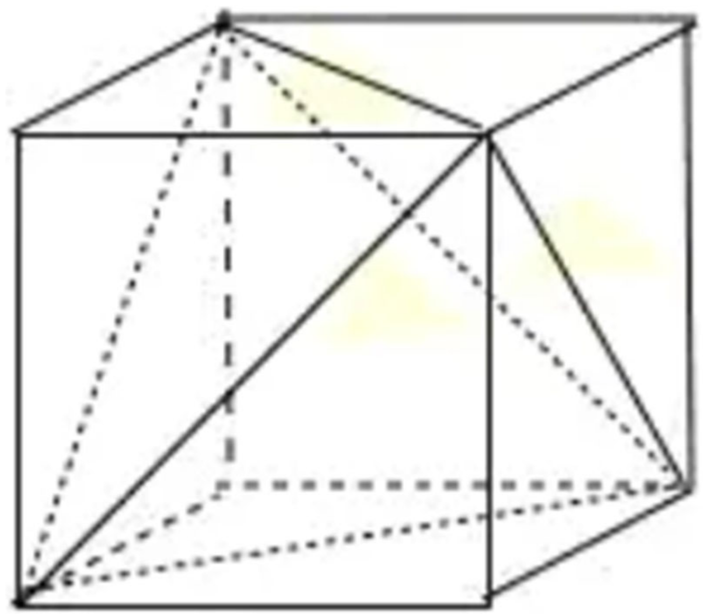
(5) $\vec{u} \times (\vec{v} \times \vec{w}) + \vec{v} \times (\vec{w} \times \vec{u}) + \vec{w} \times (\vec{u} \times \vec{v}) = \vec{0}$

7. What is the relationship between the volume of the tetrahedron defined by the vectors

$$\begin{bmatrix} a_1 \\ a_2 \\ 1 \end{bmatrix}, \quad \begin{bmatrix} b_1 \\ b_2 \\ 1 \end{bmatrix}, \quad \begin{bmatrix} c_1 \\ c_2 \\ 1 \end{bmatrix}$$

and the area of the triangle with vertices

$$\begin{bmatrix} a_1 \\ a_2 \end{bmatrix}, \quad \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}, \quad \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}?$$

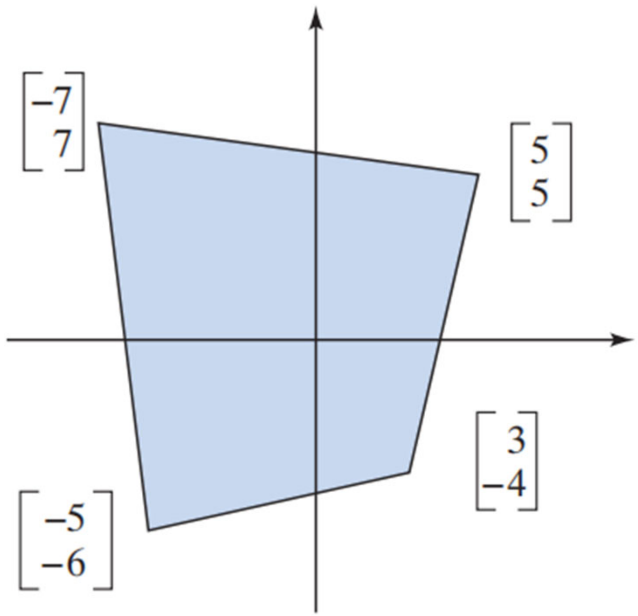


Homework

8. Find the 3-volume of the 3-parallelepiped defined by the vectors

$$\begin{bmatrix}1\\0\\0\\0\end{bmatrix}, \quad \begin{bmatrix}1\\1\\1\\1\end{bmatrix}, \quad \begin{bmatrix}1\\2\\3\\4\end{bmatrix}.$$

9. Find the area of the following region:



Homework(Optional)

6. *The cross product in* $\mathbb{R}^n$. Consider the vectors $\vec{v}_2, \vec{v}_3, \ldots, \vec{v}_n$ in $\mathbb{R}^n$. The transformation

$$T(\vec{x}) = \det\begin{bmatrix} | & | & | & & | \\ \vec{x} & \vec{v}_2 & \vec{v}_3 & \cdots & \vec{v}_n \\ | & | & | & & | \end{bmatrix}$$

is linear. Therefore, there exists a unique vector $\vec{u}$ in $\mathbb{R}^n$ such that

$$T(\vec{x}) = \vec{x} \cdot \vec{u}$$

for all $\vec{x}$ in $\mathbb{R}^n$. Compare this with Exercise 2.1.43c. This vector $\vec{u}$ is called the *cross product* of $\vec{v}_2, \vec{v}_3, \ldots, \vec{v}_n$, written as

$$\vec{u} = \vec{v}_2 \times \vec{v}_3 \times \cdots \times \vec{v}_n.$$

In other words, the cross product is defined by the fact that

$$\vec{x} \cdot (\vec{v}_2 \times \vec{v}_3 \times \cdots \times \vec{v}_n) = \det\begin{bmatrix} | & | & | & & | \\ \vec{x} & \vec{v}_2 & \vec{v}_3 & \cdots & \vec{v}_n \\ | & | & | & & | \end{bmatrix},$$

Homework(Optional)

7. for all $\vec{x}$ in $\mathbb{R}^n$. Note that the cross product in $\mathbb{R}^n$ is defined for $n-1$ vectors only. (For example, you cannot form the cross product of just two vectors in $\mathbb{R}^4$.) Since the ith component of a vector $\vec{w}$ is $\vec{e}_i \cdot \vec{w}$, we can find the cross product by components as follows:

$$
\begin{aligned}
&i\text{th component of } \vec{v}_2 \times \vec{v}_3 \times \cdots \times \vec{v}_n \\
&= \vec{e}_i \cdot (\vec{v}_2 \times \cdots \times \vec{v}_n) \\
&= \det \begin{bmatrix} | & | & | & & | \\ \vec{e}_i & \vec{v}_2 & \vec{v}_3 & \cdots & \vec{v}_n \\ | & | & | & & | \end{bmatrix}.
\end{aligned}
$$

a. When is $\vec{v}_2 \times \vec{v}_3 \times \cdots \times \vec{v}_n = \vec{0}$? Give your answer in terms of linear independence.

b. Find $\vec{e}_2 \times \vec{e}_3 \times \cdots \times \vec{e}_n$.

c. Show that $\vec{v}_2 \times \vec{v}_3 \times \cdots \times \vec{v}_n$ is orthogonal to all the vectors $\vec{v}_i$, for $i = 2, \ldots, n$.

d. What is the relationship between $\vec{v}_2 \times \vec{v}_3 \times \cdots \times \vec{v}_n$ and $\vec{v}_3 \times \vec{v}_2 \times \cdots \times \vec{v}_n$? (We swap the first two factors.)

e. Express $\det \left[\vec{v}_2 \times \vec{v}_3 \times \cdots \times \vec{v}_n \;\; \vec{v}_2 \;\; \vec{v}_3 \;\; \cdots \;\; \vec{v}_n \right]$ in terms of $\|\vec{v}_2 \times \vec{v}_3 \times \cdots \times \vec{v}_n\|$.

f. How do we know that the cross product of two vectors in $\mathbb{R}^3$, as defined here, is the same as the standard cross product in $\mathbb{R}^3$? See Definition A.9 of the Appendix.