Homework

1. (a) For matrix $A = \begin{pmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \\ 1 & 4 \end{pmatrix}$, find vectors that span the column space of A

(b)For each matrix A, find vectors that span the null space of A

(1) $A = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}$ (2) $A = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

2. Describe the solutions of the following system in parametric vector form.

$$\begin{cases} x_1 + 3x_2 - 5x_3 = 4 \\ x_1 + 4x_2 - 8x_3 = 7 \\ -3x_1 - 7x_2 + 9x_3 = -6 \end{cases}$$

Homework

3. Which of the following subsets of $\mathbf{R}^3$ are actually subspaces?

(a) The plane of vectors (b_1, b_2, b_3) with $b_1 = b_2$.

(b) The plane of vectors with $b_1 = 1$.

(c) The vectors with $b_1 b_2 b_3 = 0$.

(d) All linear combinations of $\boldsymbol{v} = (1, 4, 0)$ and $\boldsymbol{w} = (2, 2, 2)$.

(e) All vectors that satisfy $b_1 + b_2 + b_3 = 0$.

(f) All vectors with $b_1 \leq b_2 \leq b_3$.

(g). $W = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} : x + y + z = 1 \right\}$

(h). $W = \left\{ \begin{bmatrix} x + 2y + 3z \\ 4x + 5y + 6z \\ 7x + 8y + 9z \end{bmatrix} : x, y, z \text{ arbitrary constants} \right\}$

4. True or false (with a counterexample if false):

(a) The vectors $\boldsymbol{b}$ that are not in the column space $\boldsymbol{C}(A)$ form a subspace.

(b) If $\boldsymbol{C}(A)$ contains only the zero vector, then A is the zero matrix.

(c) The column space of $2A$ equals the column space of A.

(d) The column space of $A - I$ equals the column space of A (test this).

Homework(Optional)

1. How is the nullspace $\boldsymbol{N}(C)$ related to the spaces $\boldsymbol{N}(A)$ and $\boldsymbol{N}(B)$, if $C = \begin{bmatrix} A \\ B \end{bmatrix}$

2. (a) Suppose U_1, U_2 are subspaces of R^n. Prove or disprove:

    (1) The sum $U_1 + U_2 = \{\overrightarrow{u_1} + \overrightarrow{u_2} | \overrightarrow{u_1} \in U_1, \overrightarrow{u_2} \in U_2\}$ is a subspace of R^n.

    (2) The intersection $U_1 \cap U_2$ is a subspace of R^n.

    (3) The intersection $U_1 \cup U_2$ is a subspace of R^n

   (b)Suppose U_1, U_2, U_3 are subspaces of R^n. Prove or disprove:

    (1) $U_1 \cap (U_2 + U_3) = U_1 \cap U_2 + U_1 \cap U_3$

    (2) If $U_2 \subseteq U_1$, then $U_1 \cap (U_2 + U_3) = U_2 + U_1 \cap U_3$

    (3) If $U_1 + U_3 = U_2 + U_3$, then $U_1 = U_2$

3. Consider a subspace V of R^n. We define the orthogonal complement $V^\perp$ of V as the set of those $\vec{w}$ in R^n that are perpendicular to all vectors in V; in particular, $V^\perp = \{\vec{w} \in R^n | \vec{w} \cdot \vec{v} = 0 \text{ for all } \vec{v} \in V\}$. Prove or disprove: $V^\perp$ is a subspace of R^n

Homework (Optional)

4. (a) Suppose column j of B is a combination of previous columns of B. Show that column j of AB is the same combination of previous columns of AB. Then AB cannot have new pivot columns, so $\mathbf{rank}(\boldsymbol{AB}) \leq \mathbf{rank}(\boldsymbol{B})$.

   (b) Find A_1 and A_2 so that $\operatorname{rank}(A_1 B) = 1$ and $\operatorname{rank}(A_2 B) = 0$ for $B = \left[\begin{smallmatrix} 1 & 1 \\ 1 & 1 \end{smallmatrix}\right]$.

5. Problem 4 proved that $\operatorname{rank}(AB) \leq \operatorname{rank}(B)$. Then the same reasoning gives $\mathbf{rank}(\boldsymbol{B}^{\mathrm{T}} \boldsymbol{A}^{\mathrm{T}}) \leq \operatorname{rank}(A^{\mathrm{T}})$. How do you deduce that $\mathbf{rank}(\boldsymbol{AB}) \leq \mathbf{rank}\ \boldsymbol{A}$?

6. Suppose $A\boldsymbol{x} = \boldsymbol{b}$ and $C\boldsymbol{x} = \boldsymbol{b}$ have the same (complete) solutions for every $\boldsymbol{b}$. Is it true that A equals C?