Homework

1. Find a basis for the column space and null space of the matrix

(1) $\begin{pmatrix} 1 & 2 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{pmatrix}$ (2) $\begin{pmatrix} 1 & 2 & 0 & 3 & 5 \\ 0 & 0 & 1 & 4 & 6 \end{pmatrix}$ (3) $\begin{pmatrix} 1 & 1 & 5 & 1 \\ 0 & 1 & 2 & 2 \\ 0 & 1 & 2 & 3 \\ 0 & 1 & 2 & 4 \end{pmatrix}$

(4) $\begin{pmatrix} -3 & 6 & -1 & 1 & -7 \\ 1 & -2 & 2 & 3 & -1 \\ 2 & -4 & 5 & 8 & -4 \end{pmatrix}$ (5) $\begin{pmatrix} 2 & 5 & -3 & -4 & 8 \\ 4 & 7 & -4 & -3 & 9 \\ 6 & 9 & -5 & 2 & 4 \\ 0 & -9 & 6 & 5 & -6 \end{pmatrix}$

2. Which of the following are bases for R^3?

(a) $(1,2,0)$ and $(0,1,-1)$

(b) $(1,1,-1), (2,3,4), (4,1,-1), (0,1,-1)$

(c) $(1,2,2), (-1,2,1), (0,8,0)$

(d) $(1,2,2), (-1,2,1), (0,8,6)$

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3. For which value(s) of the constant k do the vectors below form a basis of $\mathbb{R}^4$?

$$\begin{bmatrix}1\\0\\0\\2\end{bmatrix}, \quad \begin{bmatrix}0\\1\\0\\3\end{bmatrix}, \quad \begin{bmatrix}0\\0\\1\\4\end{bmatrix}, \quad \begin{bmatrix}2\\3\\4\\k\end{bmatrix}$$

4. Find a basis of the subspace of $\mathbb{R}^4$ defined by the equation

$$2x_1 - x_2 + 2x_3 + 4x_4 = 0.$$

5. Consider a subspace V in $\mathbb{R}^m$ that is defined by n homogeneous linear equations:

$$\left| \begin{array}{ccccccc} a_{11}x_1 & + & a_{12}x_2 & + \cdots + & a_{1m}x_m & = & 0 \\ a_{21}x_1 & + & a_{22}x_2 & + \cdots + & a_{2m}x_m & = & 0 \\ \vdots & & \vdots & & \vdots & & \vdots \\ a_{n1}x_1 & + & a_{n2}x_2 & + \cdots + & a_{nm}x_m & = & 0 \end{array} \right| .$$

What is the relationship between the dimension of V and the quantity $m - n$? State your answer as an inequality. Explain carefully.

6.(1)Which of the matrices in this list have the same null space as matrix C?

(2)Which of the matrices in this list have the same column space as matrix C?

(3)Which of these matrices has a column space that is different from the column spaces of all the other matrices in the list?

Consider the matrices

$$C = \begin{bmatrix}1 & 1 & 1\\1 & 0 & 0\\1 & 1 & 1\end{bmatrix}, \quad H = \begin{bmatrix}1 & 0 & 1\\1 & 1 & 1\\1 & 0 & 1\end{bmatrix}$$

$$L = \begin{bmatrix}1 & 0 & 0\\1 & 0 & 0\\1 & 1 & 1\end{bmatrix}, \quad T = \begin{bmatrix}1 & 1 & 1\\0 & 1 & 0\\0 & 1 & 0\end{bmatrix}$$

$$X = \begin{bmatrix}1 & 0 & 1\\0 & 1 & 0\\1 & 0 & 1\end{bmatrix}, \quad Y = \begin{bmatrix}1 & 0 & 1\\0 & 1 & 0\\0 & 1 & 0\end{bmatrix}.$$

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7. True or False: Let A be $m \times n$ matrices. The row space of A is a subspace of R^n spanned by the rows of A

(a) If the columns of a matrix are dependent, so are the rows.

(b) The column space of a 2 by 2 matrix is the same as its row space.

(c) The column space of a 2 by 2 matrix has the same dimension as its row space.

(d) The columns of a matrix are a basis for the column space.

8. Consider an arbitrary $n \times m$ matrix A.

a. What is the relationship between the row spaces of A and $E = \text{rref}(A)$? *Hint*: Examine how the row space is affected by elementary row operations.

b. What is the relationship between the dimension of the row space of A and the rank of A?

Homework(Optional)

1. (a) Let U be the subspace of $\mathbf{R}^5$ defined by

$$U = \{(x_1, x_2, x_3, x_4, x_5) \in \mathbf{R}^5 : x_1 = 3x_2 \text{ and } x_3 = 7x_4\}.$$

Find a basis of U.

(b) Extend the basis in part (a) to a basis of $\mathbf{R}^5$.

2.Prove or disprove: Let U_1, U_2, U_3 be subspaces of R^n.

(1) If $\{\overrightarrow{v_1}, \overrightarrow{v_2}, \overrightarrow{v_3}, \overrightarrow{v_4}\}$ is a basis of U_1 and U_2 is a subspace of U_1 such that $\overrightarrow{v_1}, \overrightarrow{v_2} \in U_2$ and $\overrightarrow{v_3}, \overrightarrow{v_4} \notin U_2$, then $\{\overrightarrow{v_1}, \overrightarrow{v_2}\}$ is a basis of U_2.

(2) If U_1 is a subspace of U_2 and $dim(U_1) = dim(U_2)$, then $U_1 = U_2$.

(3) If U_1, U_2 are both 5-dimensional subspaces of R^9, then $U_1 \cap U_2 \neq \{\vec{0}\}$.

(4) $dim(U_1 + U_2) = dim(U_1) + dim(U_2) - dim(U_1 \cap U_2)$

(5) $$dim(U_1 + U_2 + U_3) = dim(U_1) + dim(U_2) + dim(U_3) - dim(U_1 \cap U_2) - dim(U_2 \cap U_3) - dim(U_1 \cap U_3) + dim(U_1 \cap U_2 \cap U_3)$$

3. Explain which of the so-called facts stated by Cramer is wrong, thus resolving the paradox.

On September 30, 1744, the Swiss mathematician Gabriel Cramer (1704–1752) wrote a remarkable letter to his countryman Leonhard Euler, concerning the issue of fitting a cubic to given points in the plane. He states two "facts" about cubics: (1) Any nine distinct points determine a unique cubic. (2) Two cubics can intersect in nine points. Cramer points out that these two statements are incompatible. If we consider two specific cubics that intersect in nine points (such as $x^3 - x = 0$ and $y^3 - y = 0$), then there is more than one cubic through these nine points, contradicting the first "fact." Something is terribly wrong here, and Cramer asks Euler, the greatest mathematician of that time, to resolve this apparent contradiction. (This issue is now known as the Cramer–Euler paradox.)

Euler worked on the problem for a while and put his answer into an article he submitted in 1747, "Sur one contradiction apparente dans la doctrine des lignes courbes" [*Mémoires de l'Académie des Sciences de Berlin*, 4 (1750): 219–233].

4.(a) Consider an $n \times n$ matrix A. Show that there exist scalars $c_0, c_1, \ldots, c_n$ (not all zero) such that the matrix $c_0 I_n + c_1 A + c_2 A^2 + \cdots + c_n A^n$ is noninvertible. *Hint*: Pick an arbitrary nonzero vector $\vec{v}$ in $\mathbb{R}^n$. Then the $n+1$ vectors $\vec{v}, A\vec{v}, A^2\vec{v}, \ldots, A^n\vec{v}$ will be linearly dependent. (Much more is true: There are scalars $c_0, c_1, \ldots, c_n$, not all zero, such that $c_0 I_n + c_1 A + c_2 A^2 + \cdots + c_n A^n = 0$. You are not asked to demonstrate this fact here.)

(b) An $n \times n$ matrix A is called *nilpotent* if $A^m = 0$ for some positive integer m. Examples are triangular matrices whose entries on the diagonal are all 0. Consider a nilpotent $n \times n$ matrix A, and choose the smallest number m such that $A^m = 0$. Pick a vector $\vec{v}$ in $\mathbb{R}^n$ such that $A^{m-1}\vec{v} \neq \vec{0}$. Show that the vectors $\vec{v}, A\vec{v}, A^2\vec{v}, \ldots, A^{m-1}\vec{v}$ are linearly independent. *Hint*: Consider a relation $c_0\vec{v} + c_1 A\vec{v} + c_2 A^2\vec{v} + \cdots + c_{m-1}A^{m-1}\vec{v} = \vec{0}$. Multiply both sides of the equation with A^{m-1} to show that $c_0 = 0$. Next, show that $c_1 = 0$, and so on.

(c) Consider a nilpotent $n \times n$ matrix A. Use the result demonstrated in Exercise (b) to show that $A^n = 0$.