Homework

1. Consider the linear transformation T from $\mathbb{R}^3$ to $\mathbb{R}^2$ with

$$T\begin{bmatrix}1\\0\\0\end{bmatrix}=\begin{bmatrix}7\\11\end{bmatrix},\quad T\begin{bmatrix}0\\1\\0\end{bmatrix}=\begin{bmatrix}6\\9\end{bmatrix},\quad \text{and}\quad T\begin{bmatrix}0\\0\\1\end{bmatrix}=\begin{bmatrix}-13\\17\end{bmatrix}.$$

Find the matrix A of T.

2. Assume T is a linear transformation. Find the standard matrix of T

(1) $T(x_1, x_2, x_3) = (x_1 - 5x_2 + 4x_3, x_2 - 6x_3)$

(2) $T: R^2 \to R^2$ rotates points (about the origin) through $\frac{3\pi}{2}$ radians (counterclockwise).

(3) T: $R^2 \to R^2$ first reflects points through the horizontal x_1-axis and then reflects points through the line $x_2 = x_1$.

3. *Find the matrices of the linear transformations from $\mathbb{R}^3$ to $\mathbb{R}^3$ given in Exercises* **a.** *through* **b.** *Some of these transformations have not been formally defined in the text. Use common sense. You may assume that all these transformations are linear.*

a. The reflection about the x–z-plane.

b. The rotation about the y-axis through an angle θ, counterclockwise as viewed from the positive y-axis.

Homework

4. Each of the linear transformations in parts (a) through (e) corresponds to one (and only one) of the matrices A through J. Match them up.

a. Scaling **b.** Shear **c.** Rotation

d. Orthogonal projection **e.** Reflection

$$A = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix}, \quad C = \begin{bmatrix} -0.6 & 0.8 \\ -0.8 & -0.6 \end{bmatrix},$$

$$D = \begin{bmatrix} 7 & 0 \\ 0 & 7 \end{bmatrix}, \quad E = \begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix}, \quad F = \begin{bmatrix} 0.6 & 0.8 \\ 0.8 & -0.6 \end{bmatrix},$$

$$G = \begin{bmatrix} 0.6 & 0.6 \\ 0.8 & 0.8 \end{bmatrix}, \quad H = \begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix}, \quad I = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix},$$

$$J = \begin{bmatrix} 0.8 & -0.6 \\ 0.6 & -0.8 \end{bmatrix}$$

5. Find a nonzero 2×2 matrix A such that $A\vec{x}$ is parallel to the vector $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$, for all $\vec{x}$ in $\mathbb{R}^2$.

6. The cross product of two vectors in $\mathbb{R}^3$ is given by

$$\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \times \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} a_2b_3 - a_3b_2 \\ a_3b_1 - a_1b_3 \\ a_1b_2 - a_2b_1 \end{bmatrix}.$$

Consider an arbitrary vector $\vec{v}$ in $\mathbb{R}^3$. Is the transformation $T(\vec{x}) = \vec{v} \times \vec{x}$ from $\mathbb{R}^3$ to $\mathbb{R}^3$ linear? If so, find its matrix in terms of the components of the vector $\vec{v}$.

7. Suppose $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_m$ are arbitrary vectors in $\mathbb{R}^n$. Consider the transformation from $\mathbb{R}^m$ to $\mathbb{R}^n$ given by

$$T\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{bmatrix} = x_1\vec{v}_1 + x_2\vec{v}_2 + \cdots + x_m\vec{v}_m.$$

Is this transformation linear? If so, find its matrix A in terms of the vectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_m$.

Homework

8. Use the formula derived in Exercise 2.1.13 to find the inverse of the rotation matrix

$$A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}.$$

Interpret the linear transformation defined by A^{-1} geometrically. Explain.

9. Find all linear transformations T from $\mathbb{R}^2$ to $\mathbb{R}^2$ such that

$$T\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \quad \text{and} \quad T\begin{bmatrix} 2 \\ 5 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}.$$

Hint: We are looking for the 2×2 matrices A such that

$$A\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \quad \text{and} \quad A\begin{bmatrix} 2 \\ 5 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}.$$

These two equations can be combined to form the matrix equation

$$A\begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix}.$$

10. Using the last exercise as a guide, justify the following statement:

Let $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_m$ be vectors in $\mathbb{R}^m$ such that the matrix

$$S = \begin{bmatrix} | & | & & | \\ \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_m \\ | & | & & | \end{bmatrix}$$

is invertible. Let $\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_m$ be arbitrary vectors in $\mathbb{R}^n$. Then there exists a unique linear transformation T from $\mathbb{R}^m$ to $\mathbb{R}^n$ such that $T(\vec{v}_i) = \vec{w}_i$, for all $i = 1, \ldots, m$. Find the matrix A of this transformation in terms of S and

$$B = \begin{bmatrix} | & | & & | \\ \vec{w}_1 & \vec{w}_2 & \cdots & \vec{w}_m \\ | & | & & | \end{bmatrix}.$$